

Subordination of Predictable Compensators

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August 9, 2015

Abstract

We consider general subordination and obtain the formula of the *subordinated* predictable compensator. An example of application is given.

Introduction

The idea of subordination (i.e. obtaining a new stochastic process by a random time change) was first introduced by Salomon Bochner in 1949 and is widely applied in the modelling of random phenomena such as stock price movements (e.g. the old Wall Street adage that "it takes volume Z to move prices X_Z "). In many applications, the *subordinated* process X_Z is discontinuous.

A central tool in the study of discontinuous process is the predictable compensator that arose from the general theory of stochastic processes [1]. The predictable compensator, which can be seen as a generalisation of the Lévy measure, gives a tractable description of the jump structure of a general stochastic process. It is an indispensable tool in many applications, for example, when performing an equivalent change of measure, an important operation in financial mathematics.

For a general time changed Markov process, the formula of the associated predictable compensator is not known. The purpose of this paper is to obtain such a formula.

Results

It is widely known that when the time of a Lévy process X is changed by an increasing Lévy process Z independent of X , the *subordinated* process X_Z is a Lévy process and the *subordinated* predictable compensator $(\mu^{X_Z})^\mathbb{P}$ of the random jump measure of X_Z can be obtained by [2,Thm 30.1]:

$$(\mu^{X_Z})^\mathbb{P}(dt, dy) = \gamma(\mu^X)^\mathbb{P}(dt, dy) + \int_{\mathbb{R}_+} P_z^X(dy) (\mu^Z)^\mathbb{P}(dt, dz), \quad (1)$$

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where $P_t^X(dy)$ is the distribution of X_t and $\gamma t = Z_t - \sum_{s \leq t} \Delta Z_s$. Extension of (1) to the case where Z is an additive subordinator has been considered in [5, Prop.1].

When X is replaced by, for example, more general diffusion process, (1) will no longer be applicable (the *subordinated* predictable compensator shall no longer be deterministic). We extend (1) to the case where X is a quasi left-continuous [1, Def.I.2.25] Markov process and Z is an increasing process and give an example of application.

Definitions and Framework

Let X and Z be two independent real-valued càdlàg processes defined on a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and Z be increasing (i.e. non-decreasing). Denote X_Z for the process obtained by a time-change of X by Z . Let \mathcal{F} be a right-continuous filtration in \mathbb{F} such that X_Z is \mathcal{F} -adapted, a non-negative random measure $(\mu^{X_Z})^\mathbb{P}$ on $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ is called the \mathcal{F} -predictable compensator of the random jump measure of X_Z [1, Thm.II.1.8.(i)] & [1, Thm.I.2.2.(i)] if for all \mathcal{F} -stopping times T and $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$

$$\mathbb{E} \sum_{t \leq T} \mathbb{1}_B(\Delta(X_Z)_t) = \mathbb{E} \int_0^T \int_B (\mu^{X_Z})^\mathbb{P}(\omega, dt, dy)$$

and that the integral process $\int_0^{t \wedge T} \int_B (\mu^{X_Z})^\mathbb{P}(\omega, ds, dy)$ is \mathcal{F} -predictable.

Denote (and respectively for Z and X_Z) $\mathcal{F}^X := (\mathcal{F}_t^X)_{t \geq 0}$ for the right-continuous and completed canonical filtration of X , $\mathcal{F}_{t-}^X := \bigvee_{s < t} \mathcal{F}_s^X$, \mathcal{P}^X for the \mathcal{F}^X -predictable σ -algebra on $\Omega \times \mathbb{R}^+$ generated by all left-continuous and \mathcal{F}^X -adapted processes and \mathbb{P}^X for the probability measure \mathbb{P} reduced to \mathcal{F}_∞^X . Denote (and respectively for Z) $(\mu^X)^\mathbb{P}$ for the \mathcal{F}^X -predictable compensator of random jump measure of X . For a $\mathcal{P}^X \otimes \mathcal{B}(\mathbb{R})$ -measurable function W , we write $W * (\mu^X)^\mathbb{P}$ for the integral process $W * (\mu^X)_t^\mathbb{P} := \int_0^t \int_{\mathbb{R}} W(\mu^X)^\mathbb{P}(\omega, ds, dy)$

Let \mathbb{D} denote the space of real-valued càdlàg functions $t \mapsto u(t)$ on \mathbb{R}^+ and \mathcal{D}_t denote the right-continuous smallest σ -algebra on \mathbb{D}_t generated by the collection of maps $\{u \mapsto u(s)\}_{s \leq t}$, $\mathcal{D}_{t-} := \bigvee_{s < t} \mathcal{D}_s$ and $\mathcal{D} := \bigvee_{t \geq 0} \mathcal{D}_t$. The product space $(\mathbb{D} \times \mathbb{D}, \mathcal{D} \otimes \mathcal{D})$ shall be denoted by $(\mathbb{D}^{\times 2}, \mathcal{D}^{\otimes 2})$. For $u, v \in \mathbb{D}$, the map $(u, v) \mapsto (u \circ v)$ is $(\mathcal{D}^{\otimes 2}, \mathcal{D})$ -measurable. We shall write $(u \circ v)(t)$ for $u(v(t))$ and $(u \circ v)(t-) = u(v(t-))$ for $\lim_{s \uparrow t} (u \circ v)(s)$ with monotonic increasing v and write

$$\mathbb{E}^X F(X., Z.) := \int_{\Omega} F(X.(\omega), Z.(\tilde{\omega})) \mathbb{P}^X(d\omega) \quad (2)$$

for all $(\mathcal{D}^{\otimes 2}, \mathcal{B}(\mathbb{R}^+))$ -measurable function F , where $\tilde{\omega}$ is any element in Ω that is being held fixed. We define the time-changed process X_Z by

$$(X_Z)_t(\omega) := X_{Z_t(\omega)}(\omega)$$

and the filtration \mathcal{F} by

$$\mathcal{F}_t := (\mathcal{F}_t^{X_Z} \bigvee \mathcal{F}_t^Z)^\mathbb{P} \quad (3)$$

and \mathcal{H} by $\mathcal{H}_t := (\mathcal{F}_t^X \bigvee \mathcal{F}_t^Z)^\mathbb{P}$. Denote \mathcal{P} and \mathcal{Q} for the \mathcal{F} (resp. \mathcal{H})-predictable σ -algebra on $\Omega \times \mathbb{R}_+$. We observe $\mathcal{P} \subset \mathcal{Q}$. We shall also denote $\mathcal{P}_{Z(\tilde{\omega})}^X$ for the predictable σ -algebra on $\Omega \times \mathbb{R}_+$ taken with respect to the filtration

$$\mathcal{F}_{Z(\tilde{\omega})}^X := (\mathcal{F}_{Z_t(\tilde{\omega})}^X)_{t \geq 0} \quad (4)$$

for every $\tilde{\omega} \in \Omega$ held fixed and call a set $N \in \Omega \times \mathbb{R}_+$ \mathbb{P} -evanescent if $\{\omega \in \Omega : \exists t \in \mathbb{R}_+, (\omega, t) \in N\}$ is \mathbb{P} -null. If X is a Markov process, we write

$$P_t^X(x, s, dy) := \mathbb{P}(X_{s+t} \in dy | X_s = x). \quad (5)$$

Proposition 1

Let $A(\omega) \geq 0$ be \mathcal{H}_t (resp. \mathcal{H}_{t-})-measurable, then there exists a $\mathcal{D}^{\otimes 2}$ -measurable $H(u, v) \geq 0$ such that $A(\omega) = H(X_*(\omega), Z_*(\omega))$ \mathbb{P} -a.s. and

$$\omega \longmapsto H(X_*(\tilde{\omega}), Z_*(\omega)) \quad (6)$$

is \mathcal{F}_t^Z (resp. \mathcal{F}_{t-}^Z)-measurable for \mathbb{P} -a.s. $\tilde{\omega} \in \Omega$ held fixed. If in addition, A is \mathcal{F}_t (resp. \mathcal{F}_{t-})-measurable, then

$$\omega \longmapsto H(X_*(\omega), Z_*(\tilde{\omega})) \quad (7)$$

is $\mathcal{F}_{Z_t(\tilde{\omega})}^X$ (resp. $\mathcal{F}_{Z_{t-}(\tilde{\omega})}^X$)-measurable for \mathbb{P} -a.s. $\tilde{\omega} \in \Omega$ held fixed.

Proof. $\Pi_t^1(\omega) := (X_*(\omega), Z_{\cdot \wedge t}(\omega))$, then Π_t^1 is a random variable defined on $(\Omega, \mathcal{H}_t, \mathbb{P})$ taking values in $(\mathbb{D}^{\times 2}, \mathcal{D}^{\otimes 2})$ and one sees $\mathcal{H}_t = ((\Pi_t^1)^{-1} \mathcal{D}^{\otimes 2})^\mathbb{P}$ by the construction of \mathcal{H}_t . If we denote $Z_{\cdot \wedge t-}$ for the map $(s \mapsto Z_{s \wedge t-}) \in \mathbb{D}$, then $\mathcal{H}_{t-} = ((\Pi_{t-}^1)^{-1} \mathcal{D}^{\otimes 2})^\mathbb{P}$ and hence if $A(\omega) = \sum a_i \mathbb{1}_{A_i}(\omega)$ for $A_i \in \mathcal{H}_t$ (resp. \mathcal{H}_{t-}) then $A(\omega) = \sum a_i \mathbb{1}_{B_i}(\Pi_t^1(\omega))$ (resp. $\mathbb{1}_{B_i}(\Pi_{t-}^1(\omega))$) \mathbb{P} -a.s. for some $B_i \in \mathcal{D}^2$. The first claim holds on simple A .

If in addition, $A_i \in \mathcal{F}_t$ (resp. \mathcal{F}_{t-}), we define a $\mathcal{D}^{\otimes 2}$ -measurable map $\Pi_t^2(u, v) := (u \circ v, v)(\cdot \wedge t)$ and $\Pi(\omega)_t := (\Pi_t^2 \circ \Pi_t^1)(\omega) = (X_{Z_{\cdot \wedge t}(\omega)}(\omega), Z_{\cdot \wedge t}(\omega))$. Observe also $(s \mapsto (u \circ v)(s \wedge t-)) \in \mathbb{D}$, one sees $\mathcal{F}_t = (\Pi_t^{-1} \mathcal{D})^\mathbb{P}$ and $\mathcal{F}_{t-} = (\Pi_{t-}^{-1} \mathcal{D})^\mathbb{P}$ hence, $A(\omega) = \sum a_i \mathbb{1}_{B_i}(\Pi_t(\omega))$ (resp. $\mathbb{1}_{B_i}(\Pi_{t-}(\omega))$) \mathbb{P} -a.s. for some $B_i \in \mathcal{D}^2$. Since $\mathbb{1}_{B_i}(X_{Z_{\cdot \wedge t}(\omega)}(\omega), Z_{\cdot \wedge t}(\omega)) = \mathbb{1}_{B_i}(\Pi_t^2(X_*(\omega), Z_{\cdot \wedge t}(\omega)))$ and that the path $s \mapsto X_{Z_{s \wedge t-}(\omega)}(\omega) \equiv X_{Z_{t-}(\omega)-}(\omega)$ for $s \geq t$, we see that the second claim also holds on simple A .

If $(H_n)_{n \geq 1}$ and H are $\mathcal{D}^{\otimes 2}$ -measurable, then $H_n \circ \Pi_t^1 \rightarrow H \circ \Pi_t^1$ \mathbb{P} -a.s. on $\Omega \Leftrightarrow H_n \rightarrow H$ $\mathbb{P} \circ (\Pi_t^1)^{-1}$ -a.s. on $\mathbb{D}^{\times 2}$. By a monotone class argument, the claims follow. \square

Proposition 2

Let $A(\omega, t) \geq 0$ be \mathcal{Q} -measurable, then there exists a $\mathcal{D}^{\otimes 2} \otimes \mathbb{R}_+$ -measurable $H((u, v), t) \geq 0$ such that $A(\omega, t) = H(X(\omega), Z(\omega), t)$ up to a \mathbb{P} -evanescent set and

$$(\omega, t) \mapsto H(X(\tilde{\omega}), Z(\omega), t) \quad (8)$$

is \mathcal{P}^Z -measurable for \mathbb{P} -a.s. $\tilde{\omega} \in \Omega$ held fixed. If in addition, A is \mathcal{P} -measurable, then

$$(\omega, t) \mapsto H(X(\omega), Z(\tilde{\omega}), t) \quad (9)$$

is $\mathcal{P}_{Z(\tilde{\omega})}^X$ -measurable for \mathbb{P} -a.s. $\tilde{\omega} \in \Omega$ held fixed.

Proof. The claims clearly holds for all \mathcal{Q} -measurable (resp. \mathcal{P} -measurable) A of the form $A_t = A_0 \mathbb{I}_{\{0\}}(t) + \sum_{i \in \mathbb{N}} A_{t_i} \mathbb{I}_{(t_i, t_{i+1}]}(t)$ for \mathcal{H}_{t_i-} (resp. \mathcal{F}_{t_i-})-measurable A_{t_i} as a direct consequence of Proposition 1. Observe also if $(H_n)_{n \geq 1}$ and H are $\mathcal{D}^{\otimes 2} \otimes \mathbb{R}_+$ -measurable then $H_n(\Pi_t^1(\omega), t) \rightarrow H(\Pi_t^1(\omega), t)$ on $\Omega \times \mathbb{R}_+$ up to a \mathbb{P} -evanescent set $\Leftrightarrow H_n((u, v), t) \rightarrow H((u, v), t)$ on $\mathbb{D}^{\times 2} \times \mathbb{R}_+$ up to a $\mathbb{P} \circ (\Pi_t^1)^{-1}$ -evanescent set. By a monotone class argument, the claims follow. \square

Theorem

Let X be a quasi left-continuous Markov process with transition kernel $P_t^X(x, s, dy)$ and Z be an increasing process independent of X . Denote X_Z the process obtained by a time-change of X by Z and $Z_t^c := Z_t - \sum_{s \leq t} \Delta Z_s$ then $(\mu^{X_Z})^\mathbb{P}(\omega, dt, dy)$ is changed as follows:

$$(\mu^X)^\mathbb{P}(\omega, dZ_t^c, dy) + \int_{\mathbb{R}_+} P_z^X(X_{Z_{t-}}, Z_{t-}, \{X_{Z_{t-}}\} + dy) (\mu^Z)^\mathbb{P}(\omega, dt, dz). \quad (10)$$

Proof. Let $A \times B \in \mathcal{P} \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$, $W := \mathbb{I}_{A \times B}$, $I := \{t \geq 0 | \Delta Z_t = 0\}$. Observe $\Delta(X_Z)_t = \Delta(X)_{Z_t}$ on I and $\Delta(X_Z)_t = X_{Z_{t-} + \Delta Z_t} - X_{Z_{t-}}$ on I^c and by (2), we can write

$$F(X_\cdot, Z_\cdot) = (W * \mu^{X_Z})_\infty = \sum_{t \in I} \mathbb{I}_A \mathbb{I}_B(\Delta(X)_{Z_t}) + \sum_{t \in I^c} \mathbb{I}_A \mathbb{I}_B(X_{Z_{t-} + \Delta Z_t} - X_{Z_{t-}}).$$

Let Z^{-1} denote the left-continuous generalized inverse of Z , by (4) & (9) put $\mathbb{I}_A = H(X_\cdot, Z_\cdot, t)$ then $(\omega, t) \mapsto H(X(\omega), Z(\tilde{\omega}), Z_t^{-1}(\tilde{\omega}))$ is \mathcal{P}^X -measurable for

\mathbb{P} -a.s. $\tilde{\omega} \in \Omega$ held fixed [1,Prop.I.2.4]. Together with the quasi left-continuity of X , [1,Thm.II.1.8] & [Cor.II.1.19], it follows $\mathbb{E} \sum_{t \in I} \mathbb{1}_A \mathbb{1}_B(\Delta(X)_{Z_t})$ (see also (2) for notation)

$$\begin{aligned}
&= \mathbb{E}^Z \mathbb{E}^X \sum_{t \in Z(I)} H(X.(\omega), Z.(\tilde{\omega}), Z_t^{-1}(\tilde{\omega})) \mathbb{1}_B(\Delta X_t) \\
&= \mathbb{E}^Z \mathbb{E}^X \int_{Z(I)} \int_{\mathbb{R}} H(X.(\omega), Z.(\tilde{\omega}), Z_t^{-1}(\tilde{\omega})) \mathbb{1}_B(y) (\mu^X)^\mathbb{P}(\omega, dt, dy) \\
&= \mathbb{E}^Z \mathbb{E}^X \int_I \int_{\mathbb{R}} H(X.(\omega), Z.(\tilde{\omega}), t) \mathbb{1}_B(y) (\mu^X)^\mathbb{P}(\omega, dZ_t(\tilde{\omega}), dy) \\
&= \mathbb{E}^Z \mathbb{E}^X \int_{\mathbb{R}_+} \int_{\mathbb{R}} H(X.(\omega), Z.(\tilde{\omega}), t) \mathbb{1}_B(y) (\mu^X)^\mathbb{P}(\omega, dZ_t^c(\tilde{\omega}), dy) \\
&= \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}} W(\omega, t, y) (\mu^X)^\mathbb{P}(\omega, dZ_t^c(\omega), dy).
\end{aligned}$$

Since X has no fixed times of discontinuity and that I^c is countable and by (8), it follows $\mathbb{E} \sum_{t \in I^c} \mathbb{1}_A \mathbb{1}_B(X_{Z_{t-} + \Delta Z_t} - X_{Z_{t-}})$

$$\begin{aligned}
&= \mathbb{E}^X \mathbb{E}^Z \sum_{t \in I^c} H \mathbb{1}_B(X_{Z_{t-} + \Delta Z_t} - X_{Z_{t-}}) \\
&= \mathbb{E}^X \mathbb{E}^Z \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} H \mathbb{1}_B(X_{Z_{t-} + z} - X_{Z_{t-}}) (\mu^Z)^\mathbb{P}(\omega, dt, dz) \\
&= \mathbb{E}^Z \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbb{E}^X [H \mathbb{1}_B(X_{Z_{t-} + z} - X_{Z_{t-}})] (\mu^Z)^\mathbb{P}(\tilde{\omega}, dt, dz).
\end{aligned}$$

By (4), (9) and [1,Prop.I.2.4], we see that for \mathbb{P} -a.s. $\tilde{\omega} \in \Omega$ held fixed, the map $\omega \mapsto H(X.(\omega), Z.(\tilde{\omega}), t)$ is $\mathcal{F}_{Z_{t-}(\tilde{\omega})}^X$ -measurable for all $t \geq 0$. Together with the Markov property of X we have $\mathbb{E}^X [H \mathbb{1}_B(X_{Z_{t-} + z} - X_{Z_{t-}})]$

$$\begin{aligned}
&= \mathbb{E}^X \left[H \mathbb{E}^X [\mathbb{1}_B(X_{Z_{t-}(\tilde{\omega}) + z} - X_{Z_{t-}(\tilde{\omega})}) | \mathcal{F}_{Z_{t-}(\tilde{\omega})}^X] \right] \\
&= \mathbb{E}^X [H P_z^X(X_{Z_{t-}(\tilde{\omega})}, Z_{t-}(\tilde{\omega}), \{X_{Z_{t-}(\tilde{\omega})}\} + B)]
\end{aligned}$$

hence $\mathbb{E} \sum_{t \in I^c} \mathbb{1}_A \mathbb{1}_B(X_{Z_{t-} + \Delta Z_t} - X_{Z_{t-}})$

$$\begin{aligned}
&= \mathbb{E}^Z \mathbb{E}^X \int_{\mathbb{R}_+ \times \mathbb{R}_+} H \int_B P_z^X(X_{Z_{t-}}, Z_{t-}, \{X_{Z_{t-}}\} + dy) (\mu^Z)^\mathbb{P}(\omega, dt, dz) \\
&= \mathbb{E} \int_{\mathbb{R}_+ \times \mathbb{R}} W \int_{\mathbb{R}_+} P_z^X(X_{Z_{t-}}, Z_{t-}, \{X_{Z_{t-}}\} + dy) (\mu^Z)^\mathbb{P}(\omega, dt, dz).
\end{aligned}$$

Define $v(\omega, dt, dy) :=$

$$(\mu^X)^\mathbb{P}(\omega, dZ_t^c, dy) + \int_{\mathbb{R}_+} P_z^X(X_{Z_{t-}}, Z_{t-}, \{X_{Z_{t-}}\} + dy) (\mu^Z)^\mathbb{P}(\omega, dt, dz)$$

then $\mathbb{E}(W * \mu^{X_Z})_\infty = \mathbb{E}(W * v)_\infty$. It is clear that $v(\omega, dt, dy)$ defines a non-negative random measure on $\mathbb{R}_+ \times \mathbb{R}$ and that $(W * v)_t$ is \mathcal{F} -predictable (3). If T is a \mathcal{F} -stopping time, put $A := \{(\omega, t) : 0 \leq t \leq T(\omega)\} \in \mathcal{P}$, (10) follows. \square

Example We calculate the compensator $(\mu^{X_Z})^\mathbb{P}$ of the random jump measure of X_Z with X and Z taken to be, respectively, a skew Brownian motion (diffusion process) and a tempered stable subordinator independent of X . The compensator of the random jump measure of Z is

$$(\mu^Z)^\mathbb{P}(dt, dz) = dt \frac{c}{z^{1+\alpha}} e^{-\lambda z} \mathbb{I}_{\{z>0\}}(dz) \quad (11)$$

for $c, \lambda > 0$ and $\alpha \in [0, 1)$. The case $\alpha = 0$ corresponds to a Gamma subordinator. By [3,(17)], the transition function of a skew Brownian motion X can be written as

$$P_t^X(x, dy) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(|y-x|)^2}{2t}} + \frac{\beta}{\text{sgn}(y)} e^{-\frac{(|y|+|x|)^2}{2t}} \right) dy \quad (12)$$

for $\beta \in [-1, 1]$. The case $\beta = 0$ corresponds to the standard Brownian motion. Using the modified Bessel function $K_v(x)$ for the integral representation

$$\int_0^\infty \frac{1}{z^{1+v}} e^{-\frac{a^2 z}{2} - \frac{b^2}{2z}} dz = 2 \left(\frac{a}{b}\right)^v K_v(ab), \quad (13)$$

the compensator formula (10) and $\phi(X_{Z_{t-}}(\omega), y) := |X_{Z_{t-}}(\omega)| + |X_{Z_{t-}}(\omega) + y|$, we obtain

$$\begin{aligned} (\mu^{X_Z})^\mathbb{P}(\omega, dt, dy) &= \frac{2c}{\sqrt{2\pi}} \left(\frac{\sqrt{2\lambda}}{|y|} \right)^{1/2+\alpha} K_{1/2+\alpha}(\sqrt{2\lambda}|y|) dt dy \\ &+ \frac{\beta 2c}{\sqrt{2\pi} \text{sgn}(X_{Z_{t-}} + y)} \left(\frac{\sqrt{2\lambda}}{\phi(X_{Z_{t-}}, y)} \right)^{1/2+\alpha} \\ &\times K_{1/2+\alpha}(\sqrt{2\lambda}\phi(X_{Z_{t-}}, y)) dt dy \end{aligned}$$

and for the Gamma case $\alpha = 0$,

$$(\mu^{X_Z})^\mathbb{P}(\omega, dt, dy) = \left(\frac{ce^{-\sqrt{2\lambda}|y|}}{|y|} + \frac{\beta}{\text{sgn}(X_{Z_{t-}} + y)} \frac{ce^{-\sqrt{2\lambda}\phi(X_{Z_{t-}}, y)}}{\phi(X_{Z_{t-}}, y)} \right) dt dy. \quad (14)$$

We see that $(\mu^{X_Z})^\mathbb{P}$ is deterministic and time-independent if and only if $\beta = 0$, in this case X_Z is a time-changed Brownian motion. If in addition, Z is a Gamma process (i.e. $\alpha = \beta = 0$) then X_Z is a Variance Gamma process [4] with Lévy measure $v(dy) = \frac{ce^{-\sqrt{2\lambda}|y|}}{|y|} dy$ and (14) reduces to

$$(\mu^{X_Z})^\mathbb{P}(\omega, dt, dy) = dt v(dy). \quad (15)$$

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